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MATTHEW O'BRIEN'S ANTICIPATION OF VECTORIAL MATHEMATICS

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SUMMARIES

In the 1840s O'Brien, a little-known mathematics professor, conceived most of the basic ideas of vector analysis, including the scalar and vector products and the Laplacian operator ∇. He introduced a notation in which he was able to discuss problems in mechanics and geometry in vector terms. In contrast nearly all his contemporaries either used quaternions or tried, ineffectively, to invent a three-dimensional algebra of vectors which had the structure of a field.

Die meisten Grundideen der Vektoranalysis einschliesslich des Skalar- und Vektorproduktes und des Laplace-Operators ∇ wurden in den Vierzigerjahren des vorigen Jahrhunderts von Matthew O'Brien, einem wenig bekannten Mathematikprofessor, erkannt. Er führte eine Schreibweise ein, welche es ihm erlaubte, Probleme der Mechanik und der Geometrie mittels Vektoren zu diskutieren. Im Gegensatz dazu benützten fast alle seiner Zeitgenossen entweder Quaternionen oder versuchten ergebnislos, eine dreidimensionale Vektoralgebra zu erfinden, welche Körperstruktur haben sollte.

1. INTRODUCTION

Vector analysis came into being quite suddenly with the publication of Gibbs' *Vector Analysis* [1881]. This work, which first appeared in two privately printed pamphlets, contains a remarkably complete account of the laws of vector algebra, including the scalar and vector products and the Laplacian operator ∇.

The appropriateness of these concepts for applications to the physical sciences is now obvious, and Gibbs' realization of their appropriateness is historically significant. Although some elements of vector analysis had been present in earlier

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work, these efforts lacked the completeness of Gibbs' work. In particular, Grassmann's *Ausdehnungslehre* [1844], in its greater generality, contained vector analysis in the sense that vector algebra could have been derived from Grassmann's system by suitable specialization of its concepts. But this did not occur.

Hamilton had shown how to construct the field of complex numbers using real number pairs [1837]; the correspondence between complex numbers and plane vectors ($xi + yj$ with (x, y)) indicated how the former may be identified with the field of real number pairs. The analogous situation in three dimensions would be for vectors in space to correspond to the elements of a field of number triples. This analogy proved irresistible to Hamilton and his contemporaries; indeed, he pursued the chimera of a field of triples for years before realizing that what he could not carry out for triples was possible for quadruples provided he abandoned commutative multiplication. In fact the two-dimensional case is a useful guide for the three-dimensional case only with respect to the additive structure.

Hamilton and his disciples--particularly P. G. Tait--believed that quaternions were the appropriate tool for solving problems in the physical sciences. Some of their followers maintained this position in opposition to "the vectorialists" until the early years of the 20th century (see [Crowe 1967; Kennedy 1979]). Looking back we can see that what makes vector analysis more appropriate for the representation of physical situations is the presence of two separate products. Although quaternions possess a multiplicative structure that is close to that of a field, it is not so well suited to applications since the quaternion product contains the scalar and vector products in combination [1]. (For if $q_m = u_m + x_m i + y_m j + z_m k = u_m + X_m$ $(m = 1, 2)$ denote quaternions, then $q_1 q_2 = u_1 u_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 + (y_1 z_2 - y_2 z_1)i + (z_1 x_2 - z_2 x_1)j + (x_1 y_2 - x_2 y_1)k = u_1 u_2 - X_1 . X_2 - X_1 \times X_2$.)

In the 1840s, long before Gibbs was working and soon after Hamilton's quaternions had appeared, an attempt to develop a vector algebra was made by Matthew O'Brien. He failed to convince his contemporaries of the value of his ideas; indeed, within a few years O'Brien himself had second (and worse) thoughts. Nevertheless his early papers contained most of the ideas requisite for an effective development of vector analysis.

2. MATTHEW O'BRIEN

Matthew O'Brien (1814-1855) is little known either to mathematicians or to historians. He was born in Ennis, Ireland, and attended Trinity College, Dublin, from 1830 to 1834 before moving to Cambridge, where he studied from 1834 to 1838. In 1840 he was elected to a Fellowship at Caius College, Cambridge. He held the post of lecturer at the Royal Military Academy at Woolwich from 1849 to 1855 and in addition was Professor of Natural Philosophy and Astronomy at King's College, London, from 1844 to 1854 [*Dictionary of Natural Biography*].

O'Brien's main interests were in the fields of astronomy and geophysics. His book, *Mathematical Tracts, Part 1* [O'Brien 1840], concerned the shape of the earth. (There is no indication that a Part 2 was ever published.) He wrote some brief papers on astronomical topics [O'Brien 1838, 1843, 1844].

O'Brien also wrote some elementary texts. In particular, those on coordinate geometry [1842b] and differential calculus [1842a] are quite good examples of the expository writing of their time. The calculus text stresses the use of the theory of limits as the foundation for the development of the definition and properties of the derivative.

Between 1846 and 1852 O'Brien wrote four papers in which ideas relating to the algebra of vectors and its applications were developed in some depth [1846, 1847a,b, 1852]. These four papers were summarized in the *Philosophical Magazine* [1847c,d,e, 1851d]. In addition there were three papers forming part of a series (never completed) which also appeared in the *Philosophical Magazine* [1851a,b,c].

In this paper I shall concentrate mainly upon the first three papers [O'Brien 1846, 1847a,b] which, I believe, contain O'Brien's most important ideas.

3. O'BRIEN'S PAPER OF 1846

O'Brien begins with an explanation of his vector notation. He writes $u = x\alpha + y\beta + z\gamma$, where α, β, γ denote three mutually perpendicular lines. (These lines form a left-handed basis, corresponding to our j, i, k, respectively.) O'Brien allows the arbitrary vector $u = x\alpha + y\beta + z\gamma$ to undergo a change of direction while remaining the same length, writing this change as $\delta u = x\delta\alpha + y\delta\beta + z\delta\gamma$, where $\delta\alpha$, $\delta\beta$, $\delta\gamma$ denote the corresponding changes in the unit vectors. By resolving $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ O'Brien shows that these changes can be expressed as $\delta\alpha = \beta c - \gamma b$, $\delta\beta = \gamma a - \alpha c$, and $\delta\gamma = \alpha b - \beta a$ and hence obtains

$$\delta u = (zb - yc)\alpha + (xc - za)\beta + (ya - xb)\gamma, \tag{4}$$

[O'Brien 1846, 416]

where a, b, c "denote arbitrary differentials." This argument is analogous to a contemporary proof of the change $\delta r = (\delta\phi e) \times v$ of a vector v due to a rotation through an infinitesimal angle $\delta\phi$ about the axis defined by e.

Multiplying (4) by λ and writing $a\lambda = x'$, $b\lambda = y'$ and $c\lambda = z'$, O'Brien obtains

$$\lambda\delta u = (zy' - yz')\alpha + (xz' - zx')\beta + (yx' - xy')\gamma. \qquad (5)$$

(This step is a normalization process.) Finally, he writes $u' = x'\alpha + y'\beta + z'\gamma$ and replaces $\lambda\delta u$ by $D_{u'}u$; hence (5) becomes

$$D_{u'}u = (zy' - yz')\alpha + (xz' - zx')\beta + (yx' - xy')\gamma. \qquad (*)$$

This is, of course, $u' \times u$ in contemporary notation.

Now O'Brien observes that it follows immediately from (*) that $D_u u' = -D_{u'}u$ and that $D_{u'+u''}u = D_{u'}u + D_{u''}u$. To indicate that the distributive law holds "we shall elevate the subscript index u' ...," says O'Brien, henceforth writing $Du'.u$ in place of $D_{u'}u$ [1846, 417]. In this notation we have the formulas

$$Du'.u = (zy' - z'y)\alpha + (xz' - x'z)\beta + (yx' - y'x)\gamma, \qquad (6)\ [21]$$

$$Du'.u = -Du.u', \qquad (7)\ [15]$$

$$D(u' + u'') = Du'.u + Du''.u, \qquad (8)\ [19]$$

where the numbers in parentheses are those used by O'Brien to identify these formulas while those in square brackets are those of the corresponding formulas in Chapter 1 of Gibbs' *Vector Analysis* [1881]. Similar formulas appear in the later account of Gibbs' lectures [1901, 62-65].

After observing that $Du'.u$ is perpendicular to both u' and u, O'Brien finds its magnitude. He denotes the magnitudes of u and u' by r and r', respectively, and the angle between them by θ. Then writing r_1 for the magnitude of $Du'.u$, he finds

$$r_1 = r'r \sin\theta. \qquad (10)\ [14]$$

Next he applies D to numerical multiples, showing that

$$D\mu u'.u = \mu Du'.u, \qquad (11)\ [16]$$

and then deduces that

$$Du'.u = x'D\alpha.u + y'D\beta.u + z'D\gamma.u. \qquad (12)$$

Further, he derives

$$D\alpha.\beta = \gamma, \quad D\beta.\gamma = \alpha, \quad D\gamma.\alpha = \beta, \qquad (13)\ [17]$$

$$D\beta.\alpha = -\gamma, \quad D\gamma.\beta = -\alpha, \quad D\alpha.\gamma = -\beta \qquad (14)\ [17]$$

and observes that

$$Du.u = 0, \qquad (15)$$

$$D\alpha.\alpha = D\beta.\beta = D\gamma.\gamma = 0. \qquad (16)$$

Finally he observes that "$Du'.u$ is generated by right-handed rotation round the axis u" [1846, 418]. The definition of $Du'.u$ as a vector is now specified in terms of magnitude, line, and sense, as well as by its components.

Before turning to the scalar product in the final sections, O'Brien considers the repeated application of the D operation, obtaining from $D\alpha.\beta = \gamma$ and $D\alpha.\gamma = -\beta$ the result, $(D\alpha)^2.\beta = -\beta$ (more clearly, $D\alpha.(D\alpha.\beta) = -\beta$), and similar results, which he sums up by remarking "hence $(D\alpha)^2$ writing before β or γ is equivalent to the sign -" [1846, 418]. In contemporary notation, $(D\alpha)^2.\beta$ is $\alpha \times (\alpha \times \beta)$. O'Brien was apparently not aware that the D operation is not associative--he never had to consider $D(D\alpha.\alpha).\beta$, i.e., $(\alpha \times \alpha) \times \beta$. Apparently he did consider the formation of new vectors by repeated binary operations as we do. As his definition and notation indicate he envisaged the D operator as analogous to taking a derivative.

Again using a variational argument O'Brien was able to derive the scalar product. He takes $u = x\alpha + y\beta + z\gamma$, assuming that x, y, and z are fixed, but that α, β, and γ vary, their variations being given by $\delta\alpha = x'\delta h$, $\delta\beta = y'\delta h$, and $\delta\gamma = z'\delta h$, respectively, "where δh is a small displacement in the direction of the line $u' = x'\alpha + y'\beta + z'\gamma$" [1846, 419]. Later he describes this in terms of a uniform expansion in the direction of u'. Since $\delta u = x\delta\alpha + y\delta\beta + z\delta\gamma$, he obtains $\delta u = (xx' + yy' + zz')\delta h$, which leads him to write $\Delta_{u'}u = xx' + yy' + zz'$ and to note that $\Delta_{u'}u = \Delta_u u'$. He states that "the operation $\Delta_{u'}$ is clearly distributive, and we shall therefore, as before, write $\Delta u'.u$ instead of $\Delta_{u'}u$." Following this he lists the properties of $\Delta u'.u$ in the formulas

$$\Delta u'.u = xx' + yy' + zz', \qquad (17) \quad [21]$$

$$\Delta u'.u = rr' \cos\theta, \qquad (18) \quad [13]$$

$$\Delta u'.u = \Delta u.u', \qquad (19) \quad [15]$$

$$\Delta(u' + u'').u = \Delta u'.u + \Delta u''.u, \qquad (20) \quad [19]$$

$$\Delta u.u = r^2, \qquad (21)$$

$$u = \alpha\Delta\alpha.u + \beta\Delta\beta.u + \gamma\Delta\gamma.u. \qquad (25)$$

[O'Brien 1846, 419]

He remarks also that if u' and u are perpendicular, then $\Delta u'.u = 0$. Further he gives formulas which are instances of the vector triple product. The first of these is

$$\Delta u'.(Du'.u) = 0; \qquad (23) \quad [24]$$

others are related to the triple product formed from an arbitrary u and two of α, β, and γ; for example $\Delta\alpha.(D\beta.u) = \Delta\gamma.u$. As the square brackets indicate, O'Brien derived most of the substantial results found in Chapter 1 of Gibbs' *Vector Analysis* [1881].

Having defined vector and scalar products and derived some properties, O'Brien shows how these ideas enable one to express conveniently and concisely certain basic results in mechanics. In Sections 17-25 of his paper he considers applications in statics. The main results, stated in Section 18 and proved in Sections 19-22, are the necessary and sufficient conditions for a system of forces (with assigned lines of action) acting upon a rigid body to be in equilibrium:

> *If the forces U, U', U'', etc., keep a rigid body at rest the six equations of equilibrium are contained in the following equations, viz.,*
>
> $$\Sigma U = 0, \qquad (28)$$
>
> $$\Sigma Du.U = 0. \qquad (29)$$
>
> [O'Brien 1846, 420]

Each force $U = X\alpha + Y\beta + Z\gamma$, etc., acts at the point $u = x\alpha + y\beta + z\gamma$, etc. Thus in modern notation (28) and (29) correspond to $\Sigma U_i = 0$ and $\Sigma r_i \times U_i = 0$. After stating (28) and (29) O'Brien expresses these equations in component form, thus explicitly exhibiting the six necessary and sufficient conditions for equilibrium. However, in the following sections the derivation (28) and (29) is carried out wholly in vector notation, which gives his proof a decidedly modern appearance.

Subsequently O'Brien finds the condition that a system of forces acting upon a rigid body and not in equilibrium should have a single resultant (rather than a couple): if $V = \Sigma U$ and $W = \Sigma Du.U$, then $Dr.V = W$, where r is the position vector of a point on the line of action of the resultant force V; thus $\Delta V.W = 0$ is the required condition.

The final sections, 25-29, concern dynamics. O'Brien begins by setting out the fundamental equations of motion of a rigid body: "Let u' be the symbol of position of any particle (δm) of a rigid body at any time (t), and U the accelerating force which acts upon m" [1846, 423]. The equations he obtains are

$$\Sigma\left(U - \frac{d^2u'}{dt^2}\right)\delta \mathrm{m} = 0, \qquad \Sigma Du'.\left(U - \frac{d^2u'}{dt^2}\right)\delta m = 0.$$

There seems to be a mistake here; for surely what is required is

$$\Sigma U - \Sigma \frac{d^2u'}{dt^2}\,\delta m = 0, \qquad \Sigma Du'.U - \Sigma Du'.\frac{d^2u'}{dt^2}\,\delta m = 0.$$

However, O'Brien's equations would be correct if his U denoted the accelerating force *per unit mass;* perhaps this is what he meant.

Writing $\bar{u}$ for "the symbol of the centre of gravity of the body" [1846, 424] and assuming $u' = \bar{u} + u$, O'Brien deduces the equations of motion,

$$m\,\frac{d^2\bar{u}}{dt^2} = \Sigma U\delta m \qquad \text{and} \qquad \Sigma Du.\frac{d^2u}{dt^2}\,\delta m = \Sigma Du.U\delta m,$$

where m is the total mass of the body. These equations correspond, in modern notation, to

$$m\,\frac{d^2\bar{r}}{dt^2} = \Sigma F \qquad \text{and} \qquad \sum\left(r \times \frac{d^2r}{dt^2}\right)\delta m = \Sigma r \times F,$$

i.e., the equation of motion of the center of mass and the equation of angular momentum about the origin.

O'Brien also writes the second of his equations in the form

$$\frac{d}{dt}\left(\Sigma Du.\frac{du}{dt}\,\delta m\right) = \Sigma Du.U\delta m, \tag{36}$$

expressing the equality of the rate of change of the moment of momentum with the sum of the moments of the impressed forces.

Subsequently he writes $du/dt = Dw.u$, where $w = w_1\alpha + w_2\beta + w_3\gamma$, and shows the left side of (36) to be $d(Aw_1\alpha + Bw_2\beta + Cw_3\gamma)/dt$, where $A = \Sigma(y^2 + z^2)\delta m$, $B = \Sigma(z^2 + x^2)\delta m$, and $C = \Sigma(x^2 + y^2)\delta m$ are the moments of inertia of the body about the coordinate axes. In a footnote O'Brien remarks that Euler's six equations follow easily from these considerations.

The final sections, 28-29, contain an application of these equations to computing the solar precession and nutation.

4. THE CAMBRIDGE PAPERS OF 1847

A few months later O'Brien published a second paper, "Contributions towards a System of Symbolical Geometry and Mechanics" [1847a]. This paper contains some new notation, an attempt to develop the elements of differential geometry using his vector notations, and an application to mechanics.

The opening sections (1-9) contain a lucid explanation of his position vector notation, much as in the earlier paper. In Section 10 he uses a limit argument to motivate the introduction of the symbol $d\varepsilon/d\theta$ for a unit vector perpendicular to a given unit vector ε [2]. He appears to have overlooked the ambiguity of sense in the definition. The D and Δ of the paper of 1846 are mentioned "both of which we shall have occasion to use hereafter."

The middle part of the paper (Sections 11-34) deals with differential geometry. In Sections 18, 19, and 21 additional notation is introduced. If u denotes a directed line (i.e., a position vector), then "du represents in magnitude and direction the element (ds) of the arc of curve." Similarly d^2u represents an infinitesimal vector pointing towards the center of curvature. Using these notations O'Brien finds the osculating and normal planes at a point on a given curve. In Sections 28 and 29 he remarks that the equations of these planes are more conveniently represented using D as well as du and d^2u.

The final sections (35-40) of this paper are not related to the earlier sections. Here O'Brien reverts to the D, Δ notation of the paper of 1846, giving a clear, concise explanation of the basic formula defining the motion of a particle acted upon by a central force. As in the earlier paper, where he illustrated the D and Δ in the context of statical equilibrium, O'Brien again shows the value of the concepts of vector analysis in applied mathematics.

Compared with his earlier paper [1846], this paper [1847a] contains very little discussion of the properties of the unit vector $d\varepsilon/d\theta$ and the d, D, and Δ operators. O'Brien was evidently more concerned with applications, particularly with applications to the geometry of curves and surfaces. The final sections (35-40) assume the reader understands the meaning of the D and Δ operators.

In the same year O'Brien published a second paper [1847b]. It begins with some "Preliminary Observations" in which O'Brien describes his object as twofold: frist, to obtain the equations of motion of a crystallized or uncrystallized medium "in their most general form ... without making any assumption as to the nature of the molecular forces," and second, "to exemplify the use of the symbolical method" [3]. Here he introduces the new notation $\mathcal{D}$ for

$$\alpha \frac{d}{dx} + \beta \frac{d}{dy} + \gamma \frac{d}{dz}$$

(compare the modern

$$i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \;)$$

[4] and expresses in a concise form the equation of motion for the case of an uncrystallized medium: $d^2v/dt^2 = \{A\mathcal{D}\Delta\mathcal{D} - B(D\mathcal{D}^2)\}v$; here A and B are constants. He also gives the corresponding equation for a crystallized medium.

In Section 8 and 9 O'Brien derives a number of formulas which correspond to formulas that are now standard in vector analysis. For example, writing $v = \alpha\xi + \beta\eta + \gamma\zeta$ he obtains

$$\Delta\mathcal{D}.v = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \qquad \text{(i.e., div } v\text{)},$$

$$\Delta\mathcal{D}.\mathcal{D} = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2, \qquad \text{(i.e., } \nabla^2\text{)}.$$

The identity $(\Delta\mathcal{D}.\mathcal{D})v - \mathcal{D}\Delta\mathcal{D}.v = -(D\mathcal{D})^2.v$ [$= -D\mathcal{D}.(D\mathcal{D}.v)$] corresponds to the modern $\nabla^2 v - \nabla(\nabla.v) = -\nabla \times (\nabla \times v)$. At the end of Section 9 he introduces the notation d_α, d_β, and d_γ to stand for the partial derivatives $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$:

> *Hence we may see immediately the meaning of the expression*
>
> $$\mathcal{D}U, \qquad or \qquad \alpha d_\alpha U + \beta d_\beta U + \gamma d_\gamma U;$$
>
> *for $\alpha d_\alpha U$ is the rate of variation of U in the direction α, affected with its proper sign of direction α, $\beta d_\beta U$ is the rate of variation in the direction β, and $\gamma d_\gamma U$ in the direction γ, each affected with its proper sign of direction. Hence compounding these rates of variation as if they were ordinary velocities, it follows that the symbolical sum $\alpha d_\alpha U + \beta d_\beta U + \gamma d_\gamma U$ expresses, in magnitude and direction, the* complete rate of variation *of the quantity U.* [O'Brien 1847b, 515]

Thus O'Brien adopts in this paper a vector notation in which the three operations, Δ, D, and $\mathcal{D}$ correspond to the scalar product, the vector product, and the Laplacian ∇, respectively. Furthermore he recognizes the utility of these operations in vector theory. To illustrate his style, I quote a few lines from Section 13:

> *$\Delta\mathcal{D}.v$ and $D\mathcal{D}.v$ may be found separately by the interpretation of a differential equation of the form*
>
> $$\frac{d^2U}{dt^2} = C\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right). \qquad (6)$$
>
> *For performing the operation $\Delta\mathcal{D}$, on both sides of (3), and putting $\Delta\mathcal{D}.v = U$ we find*
>
> $$\frac{d^2}{dt^2}(\Delta\mathcal{D}.v) = B(\Delta\mathcal{D}.\mathcal{D})\Delta\mathcal{D}.v + (A - B)(\Delta\mathcal{D}.\mathcal{D})\Delta\mathcal{D}.v;$$
>
> *or*

$$\frac{d^2U}{dt^2} = A(\Delta\mathcal{D}.\mathcal{D})U. \qquad [1847b, 517]$$

(Equation (3) referred to above is $d^2v/dt^2 = B(\Delta\mathcal{D}.\mathcal{D})v + (A - B)\mathcal{D}\Delta\mathcal{D}.v.$)

5. THE PAPERS OF 1851 AND 1852

A few years later O'Brien had second thoughts about vector analysis. He published an extended paper in the *Philosophical Transactions* [1852] and an incomplete series of papers in the *Philosophical Magazine* [1851a,b,c]. In these papers the D, Δ, and $\mathcal{D}$ notations were partially abandoned in favor of the "ordinary notation of algebra." He explained that in the earlier papers [1847a,b,c]

> *I employed a new notation to express these results, and so far obscured their meaning. I am now able to put them all into the ordinary notation of algebra without introducing anything novel in principle, or assuming any but the simplest symbolic laws.* [1851a, 395]

In [1852] O'Brien begins with two directed lines, u and v, introducing $u.v$ and $u \times v$ to denote, respectively, the "lateral, longitudinal effects of translation of u on v." In modern terms $u.v$ denotes the signed magnitude of the vector product, and $u \times v$ denotes the scalar product. Later he reintroduces D, now writing $D(u.v)$ to denote the vector product. Part 2 of the paper consists of applications to geometry, statics, dynamics, and physical optics. Near the end he introduces the symbol Ω to denote the same operator as the $\mathcal{D}$ of [O'Brien 1847a].

The main notational idea of the *Philosophical Magazine* papers [1847c,d] is to write uU to denote a couple made up of a force U acting at the point with position vector u and a force $-U$ at the origin. O'Brien then adds another force U at the origin A, observing that $U + uU$ or $(1 + u)U$ "denotes the force U acting at B" (see Figure 1). O'Brien did not investigate the algebraic consequences of this curious notation, assuming instead that the ordinary algebraic rules apply. For example, he observes that if $\gamma A + \alpha C = 0$, "it is obvious that $\gamma A = -\alpha C$" [1851a, 397]. It seems that sometimes he regarded uU as the symbol for a configuration, while at other times it denoted the moment of the force or couple. The distributive law was assumed. Despite these ambiguities and his rather casual attitude toward the manipulation of symbols, O'Brien was able to derive--to his own

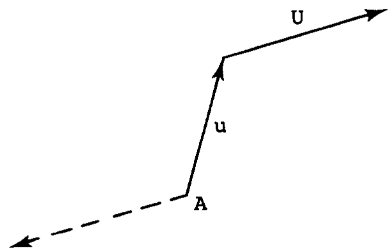

Figure 1

satisfaction at least--basic results in statics and mechanics. However, in the end he returned to the novel algebra of his previous papers. In the third paper [1851c] he introduced lines to represent forces (in [1851a,b] he had always used lower-case letters to denote lines, i.e., position vectors, and upper-case letters to denote forces), thereby obtaining again the relations $\alpha\alpha = \beta\beta = \gamma\gamma = 0$, $\beta\alpha = -\alpha\beta$, $\gamma\beta = -\beta\gamma$, and $\alpha\gamma = -\gamma\alpha$ [1851c, 123]. Thus his simple product, *uU,* in the end turned out to be the vector product, *Du.U,* in another guise.

6. CONCLUSION

O'Brien was not alone, in the 1840s and 1850s, in attempting to develop an algebra appropriate for handling problems of three-dimensional geometry and mechanics. Although little of value had been done before the appearance of Hamilton's quaternions in 1843, subsequently there was a resurgence of interest in the question.

In a footnote to the Preface of his *Lectures on Quaternions* Hamilton thanked a number of scientific contemporaries, "some of whose researches or remarks on subjects connected with quaternions have been elsewhere alluded to" [Hamilton 1853, 64]. He named no fewer than twelve British and Irish scholars: Boole, Cockle, Carmichael, Cayley, De Morgan, Donkin, Charles Graves, John Graves, Kirkman, O'Brien, Spottiswoode, and Young. However, a cursory examination of a sample of hundreds of their papers does not suggest that Hamilton could have learned much from them concerning three- and four-dimensional algebras related to geometry [5].

Most of these papers fall into two categories. Some were attempts to devise a three-dimensional algebra independent of quaternions; most of these were flawed in that their authors built into their schemes an inner multiplication and then (usually) observed that their system failed to obey the familiar laws of algebra (i.e., one or more of the field laws failed).

The second category contains papers that started with quaternions and concentrated on giving a three-dimensional geometrical interpretation of them; in some of these the i, j, k portion of the quaternion is detached, but the attempt to retain multiplication leads their authors away from a useful vector interpretation.

To show how astutely O'Brien avoided the traps into which other writers seemed inevitably to fall, I shall briefly examine papers by four contemporaries. Examples of the first kind are De Morgan's "On the Foundation of Algebra. IV. On Triple Algebra" [1849] and Charles Graves' "On a system of Triple Algebra and Its Application to the Geometry of Three Dimensions [1849].

De Morgan, inspired by Hamilton's paper on Quaternions [1844], sought to define a three-dimensional algebra closed under multiplication. He considered $a\xi + b\eta + c\zeta$, where ξ, η, and ζ are undefined base elements, and assumed that the products of pairs of the base elements satisfy $\xi^2 = \xi$, $\eta^2 = a\xi + b\eta + c\zeta$, $\zeta^2 = a\xi + c\eta + b\zeta$, $\eta\zeta = p\xi + q\eta + q\zeta$, $\zeta\xi = l\xi + m\eta + n\zeta$, $\xi\eta = l\xi + n\eta + m\zeta$. He proceeded to derive from such relations as $\xi^2\eta = \xi(\xi\eta)$, $\xi\eta^2 = \eta(\xi\eta)$, etc., (i.e., virtually the commutative and associative laws of multiplication), the conditions which must be satisfied by the coefficients. These conditions yield various cases, one of the simplest occurring when $a = -1$, $b = c = 1$, $m = 0$, and $n = b = 1$. Here we find $\xi^2 = \xi$, $\eta^2 = -\xi + \eta + \zeta$, $\zeta^2 = -\xi + \eta + \zeta$, $\eta\zeta = \xi$, $\zeta\xi = \zeta$ and $\xi\eta = \eta$. De Morgan continued to investigate the consequences of such algebras, making a brief excursion into "triple trigonometry." His investigation of an appropriate definition for the modulus yielded expressions such as $a - b - c$, $(a^2 + b^2 + c^2 + ab + ac - bc)^{1/2}$ and $(a^3 - b^3 - c^3 - 3abc)^{1/3}$. He observed that the associative law for multiplication might fail and, not surprisingly, he remarked: "I am not able to present any striking geometrical interpretation" [De Morgan 1849, 247].

Charles Graves also tried to develop a three-dimensional algebra. In "On a System of Triple Algebra" [1849] he introduced a rotation of 180° about a specified axis. Denoting this operation by s he observed that if it is applied to points a and b situated upon the x axis, then the condition $s(a) + b = 0$ implies that $a = b = 0$, $s(a) + s(b) = s(a + b) = (a + b)\ s(1)$, and $s^2(a) = a$. Introducing $s^{1/2}$ to denote a rotation through 90°, Graves found that $(1 + s(1))/2^{1/2}$ and $(1 - s(1))/\pm 2^{1/2}$ represent perpendicular directions, defining $n(1) = s^{1/2}(1 - s(1))/2^{1/2}$ to be a third unit. This permitted him to represent an arbitrary point, (x, y, z), as $x + s(y) + n(z)$. He found the product of $x + s(y) + n(z)$ and $x' + s(y') + n(z')$ to be the very cumbersome expression $xx' + yy' - zz' + s(xy' + x'y + zz') + n(x - y)z' + z(x' - y')$. It is not surprising that he did not take this system much further, and his comment, "I commend the superior power, symmetry and flexibility of Sir William Hamilton's quaternion theory" [Graves 1849, 119] seems justified.

As examples of papers of the second kind, in which quaternions were used one way or another, we consider [Donkin 1850] and [Spottiswoode 1850].

Donkin's "On the Geometrical Interpretation of Quaternions" begins with a discussion of operations: "In the present paper ... all the operations will represent *operations* and *never* concrete quantities, unless that be expressly stated" [1850, 491]. He defines + to represent "the operation of turning it [a specified straight line] round from the direction it had at first ... till it comes into the same direction again." Then - will represent $(+)^{1/2}$, i.e., half of a complete rotation, and $(+)^{\alpha}$ is a rotation of $2\pi\alpha$, which he also expressed as $\cos\theta + \sqrt{-1}\sin\theta$, where $\theta = 2\pi\alpha$. In space the axis of rotation has to be specified. Donkin writes $+_r$ to denote the operation of turning a line through a complete revolution in a plane perpendicular to a given axis r. These ideas lead to the definitions: $i = (+_x)^{1/2}$, $j = (+_y)^{1/2}$, and $k = (+_z)^{1/2}$, and to the relations $jk = i$, $ki = j$, and $ij = k$. Donkin writes $1/i$ to denote the rotation operation which reverses i. In particular, $kj = 1/i = -i$ and similarly $ik = 1/j = -j$ and $ji = 1/k = -k$. Thus Donkin's i, j, and k are identified with Hamilton's i, j, and k, and the discussion soon turns to quaternions.

Spottiswoode's "On the Geometrical Interpretation of Quaternions" [1850] cites [Donkin 1850], offering "an alternative interpretation of quaternions." Starting with quadruples of real numbers, $Q = (w, x, y, z)$, Spottiswoode uses coordinates to define equality, addition, and numerical multiplication, writing $Q - Q = 0 = (0, 0, 0, 0)$. Then he observes that $(w, x, y, z) = (w, 0, 0, 0) + (0, x, 0, 0) + (0, 0, y, 0) + (0, 0, 0, z)$, introducing the notation, Tx, $T'y$, $T''z$, for the last three terms in this identity. Finally he defines i, j, and k to be operators defined by $iQ = (-x, w, -z, y)$, $jQ = (-y, z, w, -x)$, and $kQ = (-z, -y, x, w)$. As consequences of these definitions he derives $i.iQ = j.jQ = i.j.kQ = (-w, -x, -y, -z) = -Q$, $i.kQ = -k.jQ = iQ$, $k.iQ = -i.kQ = jQ$, and $i.jQ = -j.iQ = kQ$. Spottiswoode notes that $ix = i(x, 0, 0, 0) = (0, x, 0, 0) = Tx$ and, similarly, that $jy = T'y$, $kz = T''z$. He suggests that relations such as $i^2Q = -Q$ may be "symbolically written" as $i^2 = -1$. Spottiswoode also gives formulas for the product of any two quaternions and for the product of a quaternion with its conjugate, i.e., $(w + ix + jy + kz)(w - ix - jy - kz) = w^2 + x^2 + y^2 + z^2$. The final paragraphs of the paper contain a rather unconvincing geometrical interpretation.

A significant aspect of Spottiswoode's paper is that he defines a quaternion as an ordered quadruple of real numbers, and that the operations are defined in a manner not unlike that of abstract algebra today.

Hamilton, the inventor of quaternions, also made an attempt [1846b] to develop a three-dimensional algebraic system suited to the representation of spatial geometry. Although not wholly independent of quaternions, the role of the latter was played down as much as possible: "The present paper is an attempt towards constructing a symbolical geometry, analogous in several important aspects to what is known as symbolical algebra ... and to exhibit under a new point of view his own theory ... of algebraic quaternions" [1846b, Vol. 1, 45].

Hamilton begins with a summary of the properties of directed lines. He observes that $DC = BA$ means that the lines DC and BA have the same length and direction, that BB (or AA) denotes a null line, that $CB + BA = CA$, and that $-AB = +BA$. He introduces "abridged symbols," a, b, etc., for directed lines, remarking that for addition the commutative and associative laws hold (Sections 1-5). In Section 6 he asserts that $\times$ and $\div$ must be defined so that $(b \div a) \times a = b$ will always be valid for any two lines, a and b, and that, further, $a \div a = 1$ and $(a - a) \div a = 0$ must hold.

> *The formula* $[(b \div a) \times a = b]$ *will then* express nothing respecting those lines *themselves which can serve to distinguish them from any other lines in space; but will furnish* a symbolic condition, *which we must satisfy by the* general *interpretation of a geometrical quotient, and of the* operation of multiplying a line *by such a quotient.* [Hamilton 1846b, Vol. 1, 52]

In Section 7 Hamilton gives a little more specific information about the concept of the quotient of two lines. He considers two special cases: frist, when the lines are parallel, and second, when they are perpendicular. If a is parallel to c, then $c = \alpha a$, for some scalar α; by $c \div a$ Hamilton understands the *scalar* α. If a is perpendicular to c, then he calls the quotient $c \div a$ a *vector quotient* and uses italic letters to denote such objects. He says that

> *a vector quotient* $c \div a = \mathfrak{a}$ *may be regarded as denoting the* relative length and relative direction *(depending on the plane and hand) of two perpendicular lines a, c; or as indicating* in what ratio *the length of one line* $\mathfrak{a}$ *must be altered (if at all) in order to become equal to the length of another line c, and also* round what axis, *perpendicular to both those two rectangular lines, the direction of the divisor line a must be caused or conceived to turn, right-handedly, through a right angle, in order to attain the original direction of the dividend line c.* [1846b, Vol. 1, 55-56]

Thus the quotient vector is conceived as an operator which makes a line into another line. The general case of a vector quotient is then defined by resolving the dividend into lines parallel and perpendicular (respectively) to the divisor. Thus, to define $e \div a$, write $e = c + b$, where b is parallel to a, and c is perpendicular to a; then $e \div a = e \div c + e \div b$. Hamilton writes $V(e \div a) = c \div a$ and $S(e \div a) = b \div a$, calling these the scalar and vector parts of the vector quotient. It is apparent that although quaternions have not been mentioned, they are not, in reality, far away.

Later in the paper Hamilton introduces additional operators on vector quotients; again, these strongly remind us of quaternions. For example, in Section 40 he defines

$$T\,\frac{b}{a} = \left[\left(S\frac{b}{a}\right)^2 - \left(V\frac{b}{a}\right)^2\right]^{1/2}.$$

Here b/a replaces more cumbersome $b \div a$. Many of the later sections of this paper are concerned with applications of these ideas to geometry--frequently to problems concerning cones and ellipsoids [1846b, Vol. 4, 84-85].

These papers written by four of O'Brien's contemporaries are by no means the worst of their kind; they show the difficulties that most writers created for themselves--particularly when they attempted to include multiplication and division of vectors in their systems. It is surely significant that O'Brien avoided this difficulty. Today we know that it is impossible to have a field (or even a division ring) structure in a three-dimensional algebra. The only way to develop a *useful* three-dimensional algebra is to avoid introducing multiplication in the hope of attaining a field-like structure.

M. J. Crowe has discussed O'Brien's work on vectors [1967, Chap. 7, IX], concentrating on [O'Brien 1852]. Crowe has observed that "O'Brien should be viewed as a forerunner of Gibbs and Heaviside, but he did not anticipate them in the construction of the modern system. His system was rather primitive compared with theirs" [1967, 100].

A point upon which Crowe does not do justice to O'Brien concerns the quantity $u.v$: "He also failed, up to this point in the paper, to state whether u,v and $u \times v$ were numerical magnitudes or directed magnitudes" [Crowe 1967, 98]. It seems clear to me that O'Brien defined u,v and $u \times v$ to be numerical magnitudes, later defining $D(u,v)$ to be a directed magnitude (our $\underset{\sim}{u} \times v$) having u,v as its numerical magnitude.

Moreover, in his Cambridge papers [1846, 1847a,b] O'Brien came close to constructing the modern system of vector algebra in much the way that Gibbs gave it a generation later.

In the papers of 1851 and 1852 O'Brien's second thoughts were certainly less close to the modern system. Perhaps here Crowe's evaluation is perfectly fair. O'Brien's notation in the Cambridge papers may have been a little cumbersome; yet his successful application of vector methods to various topics of geometry and mechanics shows that it would serve. However, his ideas appear to have had little, if any, influence. One might say that he convinced no one; bearing in mind the near repudiation of vectors in the later papers, he did not even succeed in convincing himself.

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NOTES

1. However, some writers have found quaternions useful in the discussion of special relativity (see, e.g., [Silberstein 1924]).

2. O'Brien considers $AP' = r'\varepsilon'$, $AP = r\varepsilon$ (see Figure 2, which reproduces that of his paper), then "if $r' = r$ and $\varepsilon' - \varepsilon$ is indefinitely small, the expression $[r'\varepsilon' - r\varepsilon]$ becomes $rd\varepsilon$." But the length of $rd\varepsilon$ is $rd\theta$ "assuming $d\theta$ = angle PAP' so the direction unit is $d\varepsilon/d\theta$" [O'Brien 1847d, 498-499].

3. By "crystallized" and "uncrystallized" O'Brien meant "isotropic" and "anisotropic."

4. Hamilton introduced the symbol $\triangleleft$ for this vector operator before O'Brien--see [Hamilton 1846a].

5. The work of British algebraists in the middle of the 19th century is discussed in [Koppelman 1971]. In particular, Koppelman refers to Donkin, Charles Graves, Spottiswoode, and, of course, Hamilton.

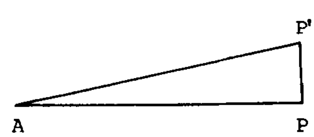

Figure 2

REFERENCES

(Where two dates are given, the first is the date the paper was read and the second is the publication date.)

Crowe, M. J. 1967. *A history of vector analysis*. Notre Dame.

De Morgan, A. 1844 (1849). On the foundation of algebra. IV. On triple algebra. *Transactions of the Cambridge Philosophical Society* 8, 241-254.

Donkin, W. F. 1850. On the geometrical interpretation of quaternions. *Philosophical Magazine* (3) 36, 489-503.

Dictionary of National Biography. Matthew O'Brien, Vol. 14, p. 764.

Gibbs, J. W. 1881. Vector analysis (privately printed; reprinted in Gibbs' *Collected scientific papers*, Vol. 2, pp. 17-90).

——— 1901. *Vector analysis*. New Haven. Reprinted in 1922 and 1960. E. B. Wilson, Ed.

Grassmann, H. G. 1844. *Die Lineale Ausdehnungslehre ein Neuer Zweig der Mathematik*. Leipzig.

Graves, C. 1849. On a system of triple algebra and its application to the geometry of three dimensions. *Philosophical Magazine* (3) 34, 119-126.

Hamilton, W. R. 1837. Theory of conjugate functions or algebraic couples. *Transactions of the Royal Irish Academy* 17, Pt. 2, 293-422 (*Mathematical Papers* 3, 3-96).

——— 1844. (1849). On quaternions. *Philosophical Magazine* (3) 34, 241-246.

——— 1846a (1847). On a proof of Pascal's theorem. *Proceedings of the Royal Irish Academy* 3, 273-292 (*Mathematical Papers* 3, 367-377).

——— 1846b- 1849. On symbolical geometry. *Cambridge and Dublin Mathematical Journal* 1 (1846), 45-57, 137-154, 256-263; 2 (1847), 47-52, 130-133, 204-209; 3 (1848), 68-84, 220-225; 4 (1849), 84-89, 105-118.

——— 1853. *Lectures on quaternions*. Dublin.

——— 1967. *The Mathematical papers*, Vol. 3. Cambridge.

Kennedy, H. 1979. James Mills Peirce and the cult of quaternions. *Historia Mathematica* 6, 423-429.

Koppelman, E. 1971. The calculus of operations and the rise of abstract algebra. *Archive for History of Exact Sciences* 8, 155-242.

O'Brien, M. 1838 (1839). On the variation of the longitude of the perihelion in the planetary theory. *Cambridge Mathematical Journal* 1, 165-166.

——— 1840. *Mathematical tracts*, Pt. 1. Cambridge.

——— 1842a. *An elementary treatise on the differential calculus in which the method of limits is exclusively made use of*. Cambridge.

——— 1842b (1844). *A treatise on plane coordinate geometry; or the application of the method of coordinates to the solution of problems in plane geometry.* Cambridge.

——— 1843. On certain formulae made use of in physical astronomy. *Cambridge Mathematical Journal* 3, 249-257.

——— 1844. On the lunar theory. *Cambridge Mathematical Journal* 4, 49-59.

——— 1846 (1849). On a new notation for expressing various conditions and equations in geometry, mechanics, and astronomy. *Transactions of the Cambridge Philosophical Society* 8, 415-428.

——— 1847a (1849). Contributions towards a system of symbolical geometry and mechanics. *Transactions of the Cambridge Philosophical Society* 8, 497-507.

——— 1847b. On the symbolic equation of vibratory motion of an elastic medium whether crystallized or uncrystallized. *Transactions of the Cambridge Philosophical Society* 8, 508-523.

——— 1847c. On a new notation for expressing various conditions and equations in geometry, mechanics and astronomy. *Philosophical Magazine* (3) 31, 134-136.

——— 1847d. Contributions towards a system of symbolical geometry and mechanics. *Philosophical Magazine* (3) 31, 139-143.

——— 1847e. On the symbolical equation of vibratory motion of an elastic medium whether crystallized or uncrystallized. *Philosophical Magazine* (3) 31, 376-380.

——— 1851a. On the interpretation of the product of a line and a force. *Philosophical Magazine* (4) 1, 394-397.

——— 1851b. On symbolical statics. *Philosophical Magazine* (4) 1, 491-495.

——— 1851c. On symbolic mechanics. *Philosophical Magazine* (4) 2, 121-125.

——— 1851d. Researches in symbolical physics. On the translation of a directed magnitude as symbolised by a product. *Philosophical Magazine* (4) 2, 149-150.

——— 1852 . On symbolic forms derived from the conception of the translation of a directed magnitude. *Philosophical Transactions of the Royal Society* 142, 161-206.

Silberstein, L. 1924. *The theory of relativity.* 2nd ed. London.

Spottiswoode, W. 1850. On the geometrical interpretation of quaternions. *Philosophical Magazine* (3) 37, 108-118.